Special Relativity

FIZIKA SPhO Training

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1 Notes

In Special Relativity, you will now learn that all of classical physics you have studied is incorrect. Or, perhaps just *incomplete.*

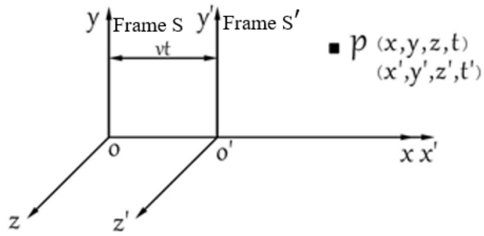
1.1 Galilean Relativity

In kinematics, we discussed frames of reference, and introduced the idea of position being a relative quantity that depends on the observer's frame. All the physics you have learnt so far follows what we call **Galilean Relativity.**

1.1.1 The Standard Configuration

Technically, a frame of reference consists of spatial x, y and z-axes and time t. However, it is difficult to track 4 coordinates at once. Thus, in Physics Olympiad, we are usually only concerned about x and t.

We shall use the following **standard configuration:**



The standard configuration obeys the following:

1. Frame S' moves at a constant velocity v in the $+x$ direction relative to frame S along the x and x'-axes.
2. Frame S and S' have their origins O and O' coincide at $t = t' = 0$.

1.1.2 Galilean Transformations

Classically, between frame S and S', the following transformations hold:

$$x' = x - vt \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$t' = t \tag{4}$$

Equations (1) to (4) are known as the **Galilean transformations.** These are perfectly valid when you are dealing with everyday speeds $v \ll c$. In fact, you have been using them in classical mechanics (think relative velocity)!

Equation (4) has important consequences. It implies that **time is absolute** in Galilean relativity – that all observers see the same time interval.

1.2 Einstein's Postulates

In the early 20th century, Einstein made the following postulates.

1.2.1 The Principle of Relativity

Statement: The laws of physics, such as mechanics and electromagnetism, must be identical and hold in all inertial frames of reference.

This suggests that there is no "special" inertial frame that is "correct" or the "best". All inertial frames are equally valid!

1.2.2 The Principle of Invariant Light Speed

Statement: The speed of light in a vacuum is the same in all inertial frames, and is constant at c.

Accepting this principle automatically means rejecting the Galilean Transformations in Section 1.1.2, since the Galilean Transformations allow for the speed of light to be different from c.

1.2.3 Significance of the Postulates

Why are these postulates important? The fundamental reason is that we either:

1. Accept Einstein's postulates, but reject Galilean relativity
2. Reject Einstein's postulates, and continue accepting Galilean relativity

because the two directly contradict each other.

For physicists *at that time*, the answer was clear – Option 2. Everyone all agreed that rejecting the notion that time was absolute was ridiculous; after all, how could you moving in a car and myself remaining stationary measure different time intervals?

However, it turns out that you will, indeed, measure *different* time intervals. Einstein's seemingly ridiculous postulates were actually proven to be right.

If you are interested, you may read up on the Michelson-Morley Experiment. It disproved the existence of *ether*, which was a fictional medium that physicists thought light travelled through, paving way for the principle of invariant light speed.

1.3 The Fundamentals of Special Relativity

There are four fundamentals of Special Relativity that you need to know. The proofs are omitted for conciseness. Hereby, we shall also define a **reference frame** as something consisting of a lattice of space and time coordinates.

1.3.1 The Invariant Interval

The **invariant interval,** $\mathrm{d}s$, is defined as

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 - (c\,\mathrm{d}t)^2 \tag{5}$$

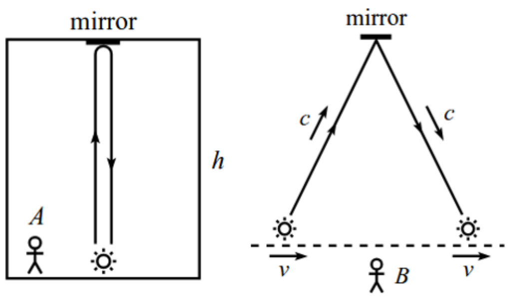
This quantity is invariant across all inertial frames (i.e. $\mathrm{d}s = \mathrm{d}s'$).

Technically, knowing this is enough to solve *all* relativistic kinematics problems. However, this is usually not the most efficient way.

Remark. Some books may define the invariant interval as the negative of Equation (5). This is purely just a sign convention thing – as long as you are consistent, physics won't break itself.

1.3.2 Time Dilation

Consider the following thought experiment. A light beam is shone from a light source on the floor of a train of height h moving with speed v, and strikes a mirror directly above. Two observers, A and B, are at rest in the train and at rest on the ground respectively.



Observers A and B see the time intervals

$$t_A = \frac{2h}{c},\ t_B = \frac{2h}{\sqrt{c^2 - v^2}} \tag{6}$$

We could be interested in the ratios of these times:

$$\frac{t_B}{t_A} = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} \tag{7}$$

At this juncture, we shall introduce two quantities, β and γ, which are ubiquitous in special relativity:

$$\beta := \frac{v}{c},\ 0 \leq |\beta| \leq 1 \tag{8}$$

$$\gamma := \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} = \frac{1}{\sqrt{1 - \beta^2}},\ \gamma \geq 1 \tag{9}$$

This allows us to write Equation (7) in the form

$$t_{\mathrm{B}} = \gamma t_A \tag{10}$$

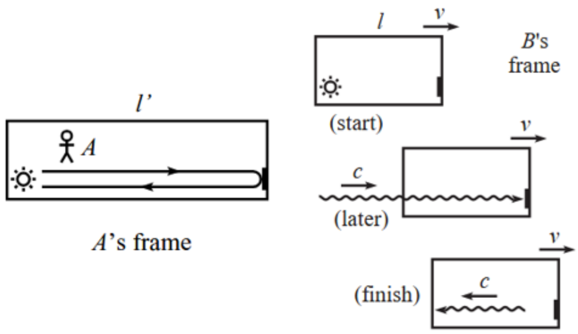
which is referred to as **time dilation.**

Here, t_A is referred to as the **proper time,** which is also the shortest possible time interval between two events. The proper time occurs in the frame whereby the two events happen at **the same position.** (The two events here are the light coming out of the source and the light going back to the source.) All other time intervals in all other frames are dilated, and longer than the proper time.

Remark. It is very important for you to identify which frame sees the proper time, so that your factors of γ are correct! Always remember how proper time is defined. In some scenarios, there might not even be a frame observing the proper time.

1.3.3 Length Contraction

Consider another thought experiment. A light beam is shone from a light source at the left end of a train of length l' as measured by a person in the train, and strikes a mirror at the right end, before getting reflected back. There is another observer on the ground who sees the train moving at speed v, and measures the length of the train to be l.



Observer A sees the time interval

$$t_A = \frac{2l'}{c} \tag{11}$$

For observer B, the relative speed of the light and mirror is first $c - v$. After reflection, the relative speed with the left end is $c + v$. Hence, he sees the time interval

$$t_B = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2l\gamma^2}{c} \tag{12}$$

By definition, observer A sees the proper time interval, and observer B sees the dilated time. Hence,

$$t_B = \gamma t_A \quad \Longrightarrow \quad \frac{2l\gamma^2}{c} = \frac{2l'\gamma}{c} \quad \Longrightarrow \quad l = \frac{l'}{\gamma} \tag{13}$$

Equation (13) is referred to as **length contraction.**

Here, l' is referred to as the **proper length,** which is also the longest possible length measured. The proper length occurs in **the rest frame** of the object being measured. (In this case, the proper length of the train is in the rest frame of the train, which is observer A's frame.) All other lengths in all other frames are contracted, and shorter than the proper length.

Remark. Time dilation and length contraction are actually equivalent! You should not be using both simultaneously for the same situation, as that will "double-count" factors of γ.

Remark. Length contraction only applies **parallel to the direction of motion!** Lengths that are perpendicular to the direction of motion are not contracted.

Example 1.1 (Ricardo, modified)**.** A muon whose half-life is $2\,\mu$s as measured by an observer at rest relative to the muon, is moving with a velocity of $0.9c$. If a large burst of such muons is produced at a certain point in the atmosphere, but only 1% reach the Earth's surface, estimate the height from which the burst originated, using (i) time dilation (ii) length contraction.

This example will be worked out in more detail to show you the thought process.

(i) Using the first-order decay equation (from nuclear physics), we have

$$N(t) = N_0\, e^{-\lambda t}, \;\; \lambda = \frac{\ln 2}{t_{\frac{1}{2}}}$$

Hence, in the muon's frame, we have

$$0.01 = \left(\frac{1}{2}\right)^{\frac{t}{2\mu\text{s}}} \quad \Longrightarrow \quad t = 13.3\mu\text{s}$$

The γ factor is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.9^2}} = 2.29$$

As the events of the muon burst and the decay of the muon happen at the same position in the muon's frame (at the position of the muon itself), the muon sees the proper time interval. Hence, the Earth sees the dilated time:

$$t_{\mathrm{E}} = \gamma t$$

The height where the burst originated is hence

$$h = v t_{\mathrm{E}} = v\gamma t = 8230\,\mathrm{m}$$

(ii) If we use length contraction instead, the proper length is in the Earth's frame. Hence, the muon sees the contracted length:

$$h' = vt$$

Hence, the proper length (the height where the burst originated in the Earth's frame) is

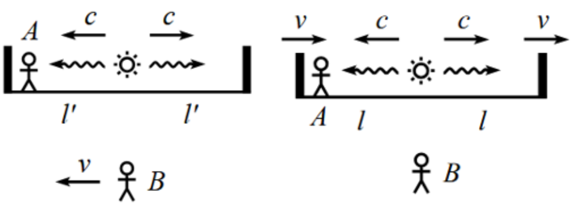
$$h = \gamma h' = \gamma v t = 8230\,\mathrm{m}$$

This example shows you that time dilation and length contraction are used exclusively of one another. If you applied both at the same time, you will get wrong answers!

1.3.4 Loss of Simultaneity

We call two (or more) events **simultaneous** if they occur at the same time in a given frame.

Consider another thought experiment. A point light source is placed at the centre of a train of length $2l'$, measured in its rest frame. Observer A is in the train, while observer B is on the ground and is moving at speed v relative to the train, measuring the train to have length $2l$.



Clearly, observer A will see that light hits both ends of the train at the same time:

$$t_{A,\text{left}} = t_{A,\text{right}} = \frac{l'}{c} \tag{14}$$

However, observer B sees the ends of the train moving. The relevant relative speeds between the photon and the ends of the train are $c+v$ and $c-v$ for the left and right ends respectively. As such, the times taken are

$$t_{B,\text{left}} = \frac{l}{c+v},\ t_{B,\text{right}} = \frac{l}{c-v} \tag{15}$$

The two events are hence no longer simultaneous in observer B's frame! They differ by the time

$$\Delta t = t_{B,\text{right}} - t_{B,\text{left}} = \frac{2lv}{c^2-v^2} = \frac{2lv\gamma^2}{c^2} \tag{16}$$

The point here is that **events that are simultaneous in one frame will no longer be simultaneous in another frame.**

1.4 Relativistic Kinematics

Beyond these basic results, there are more mathematical tools to help us study relativistic kinematics.

1.4.1 Lorentz Transformations

The **Lorentz Transformations** (sometimes called LT) are a set of linear transformations used to transform the spacetime coordinates between two frames, usually (ct, x, y, z) in the S frame and (ct', x', y', z') in the S' frame. For Physics Olympiad, you will basically never have to deal with the y and z directions, so we will ignore them.

Remark. You might have noticed that time is not written as t (or t'), but is written as ct (or ct'). This is more convenient for us as we can express all the coordinates in units of distance.

When going from S to S', the LT is

$$x' = \gamma\left(x - vt\right),\ t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{17}$$

or, it can be written cleanly in matrix form as

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{18}$$

And, when going from S' to S, the inverse LT is

$$x = \gamma\left(x' + vt'\right),\ t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{19}$$

or, it can be written cleanly in matrix form as

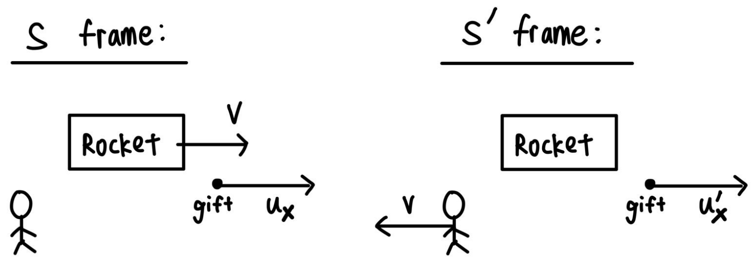
$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{20}$$

Remark. The direction of relative velocity of S and S' is important here! As per the standard configuration, this assumes S' travels with velocity v in the $+x$ direction relative to S. If S' is going in the $-x$ direction instead, then $\beta < 0$.

The LT also works for spacetime intervals – you can replace all the x and t (and x' and t') with Δx and Δt (and $\Delta x'$ and $\Delta t'$) and it will still work.

1.4.2 Relativistic Velocity Addition

Consider the following set-up. A rocket (frame S') moves at a speed v in the $+x$ direction relative to you (frame S). You notice that it ejects out a gift at a speed u_{x} in your frame, in the $+x$ direction too. What is the speed of the gift relative to the rocket? (Assume all speeds are relativistic.)



If there was no "new" way to add speeds, then the Galilean result $u_\mathrm{x} = v + u'_\mathrm{x}$ could exceed c. Clearly, this implies there must be a relativistic way of adding speeds to avoid this.

From the LT,

$$x' = \gamma\left(x - vt\right) \implies \frac{\mathrm{d}x'}{\mathrm{d}t} = \gamma\left(\frac{\mathrm{d}x}{\mathrm{d}t} - v\right) = \gamma\left(u_\mathrm{x} - v\right) \tag{21}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{\mathrm{d}t'}{\mathrm{d}t} = \gamma\left(1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \gamma\left(1 - \frac{u_\mathrm{x}v}{c^2}\right) \tag{22}$$

The speed of the gift relative to the rocket is hence

$$u'_\mathrm{x} = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}x'}{\mathrm{d}t}}{\frac{\mathrm{d}t'}{\mathrm{d}t}} = \frac{\gamma\left(u_\mathrm{x} - v\right)}{\gamma\left(1 - \frac{u_\mathrm{x}v}{c^2}\right)} = \frac{u_\mathrm{x} - v}{1 - \frac{u_\mathrm{x}v}{c^2}} \tag{23}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Equation (23) is referred to as the **relativistic velocity addition formula.** Clearly, if you set $u_\mathrm{x} = c$ (which is the maximum possible case), you obtain $u'_\mathrm{x} = c$, which makes sense given the invariance of c in all frames.

Remark. One common mistake is forgetting that *time* is also transformed, and leaving out Equation (22). Remember that time is no longer absolute, and is now relative!

Remark. When applying Equation (23), pay attention to which velocities are referring to which objects, and also the signs of u_x and v. Always make sure to sanity-check your answer (for instance, ensure you indeed obtain $u'_\mathrm{x} < u_\mathrm{x}$ if u_x and v are in the same direction).

Equation (23) only holds in the x-direction. Interestingly, if you are dealing with velocities in the y and z directions, they are transformed as well because time is still transformed (even though y and z positions aren't)! They obey the following:

$$u'_\mathrm{y} = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}y'}{\mathrm{d}t}}{\frac{\mathrm{d}t'}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}t'}{\mathrm{d}t}} = \frac{u_\mathrm{y}}{\gamma\left(1 - \frac{u_\mathrm{x}v}{c^2}\right)} \tag{24}$$

$$u'_\mathrm{z} = \frac{\mathrm{d}z'}{\mathrm{d}t'} = \frac{\frac{\mathrm{d}z'}{\mathrm{d}t}}{\frac{\mathrm{d}t'}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}z}{\mathrm{d}t}}{\frac{\mathrm{d}t'}{\mathrm{d}t}} = \frac{u_\mathrm{z}}{\gamma\left(1 - \frac{u_\mathrm{x}v}{c^2}\right)} \tag{25}$$

Example 1.2. Consider frame S' moving at speed v in the $+x$-direction relative to frame S, as per the standard configuration. A beam of light travels in the $+y$-direction in frame S. Verify that light does not slow down in frame S'.

We simply use relativistic velocity addition for the x and y-directions, noting that $u_\mathrm{x} = 0, u_\mathrm{y} = c$ (since light travels in purely the y-direction in frame S):

$$u'_\mathrm{x} = \frac{u_\mathrm{x} - v}{1 - \frac{u_\mathrm{x}v}{c^2}} = \frac{0 - v}{1 - 0} = -v$$

$$u'_\mathrm{y} = \frac{u_\mathrm{y}}{\gamma\left(1 - \frac{u_\mathrm{x}v}{c^2}\right)} = \frac{c}{\gamma\left(1 - 0\right)} = \frac{c}{\gamma}$$

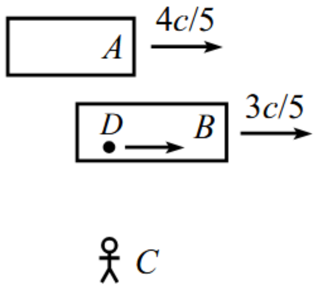
Hence, the speed of light in frame S' is

$$u' = \sqrt{u'^2_\mathrm{x} + u'^2_\mathrm{y}} = \sqrt{\left(\frac{c}{\gamma}\right)^2 + (-v)^2} = \sqrt{c^2\left(1 - \frac{v^2}{c^2}\right) + v^2} = \sqrt{c^2} = c$$

Even though its direction has changed, the speed of light is invariant across frames, as expected.

The next example will tie together the concepts we have learnt thus far, and it is very important for you to understand it thoroughly.

Example 1.3 (Morin, modified)**.** Two trains, A and B, each have proper length L and move in the same direction. A's speed is $\frac{4c}{5}$, and B's speed is $\frac{3c}{5}$. A starts behind B. (i) How long, as viewed by person C on the ground, does it take for A to overtake B? (By this, we mean the time between the front of A passing the back of B, and the back of A passing the front of B.) What about when viewed by A and viewed by B? (ii) Let event E_1 be "the front of A passing the back of B", and let event E_2 be "the back of A passing the front of B". Person D walks at a constant speed from the back of B to the front, such that he coincides with both events E_1 and E_2 in position. How long does the "overtaking" process take, as viewed by D? (iii) For all 4 observers, verify that the invariant interval is the same.



(i) Person C sees the following γ factors for A and B respectively:

$$\gamma_{A,C} = \frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}} = \frac{5}{3},\ \gamma_{B,C} = \frac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{5}{4}$$

As such, by length contraction, he sees the following lengths for A and B:

$$L_{A,C} = \frac{L}{\gamma_{A,C}} = \frac{3}{5}L,\ L_{B,C} = \frac{L}{\gamma_{B,C}} = \frac{4}{5}L$$

The relative speed of A and B in C's frame is given by a simple difference:

$$\Delta v_C = v_{A,C} - v_{B,C} = \frac{4c}{5} - \frac{3c}{5} = \frac{c}{5}$$

As such, the time interval for the overtaking process in C's frame is

$$\Delta t_C = \frac{L_{A,C} + L_{B,C}}{\Delta v_C} = \frac{\frac{3}{5}L + \frac{4}{5}L}{\frac{c}{5}} = \frac{7L}{c}$$

Remark. There is **no relativistic velocity addition** here, since both speeds $\frac{3c}{5}$ and $\frac{4c}{5}$ are being measured with respect to the same frame (C's frame).

For A's frame, by relativistic velocity addition, he sees B moving at

$$v_{B,A} = \frac{\frac{3}{5} - \frac{4}{5}}{1-\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)}c = -\frac{5c}{13}$$

The negative sign means that B moves to the left in A's frame. The associated γ factor and hence the contracted length of B in this frame are

$$\gamma_{B,A} = \frac{1}{\sqrt{1-\left(-\frac{5}{13}\right)^2}} = \frac{13}{12} \quad \Longrightarrow \quad L_{B,A} = \frac{L}{\gamma_{B,A}} = \frac{12}{13}L$$

Hence, the time interval for the overtaking process in A's frame is

$$\Delta t_A = \frac{L_{A,A} + L_{B,A}}{|v_{B,A}|} = \frac{L + \frac{12}{13}L}{\frac{5c}{13}} = \frac{5L}{c}$$

You can work out something similar for B's frame. Alternatively, you could realise that the situation is symmetrical to A's frame, and hence the time interval in B's frame is also

$$\Delta t_B = \frac{L_{A,B} + L_{B,B}}{v_{A,B}} = \frac{5L}{c} = \Delta t_A$$

(ii) Consider going into B's frame. The two events are separated by a distance of $\Delta x = L$ and a time interval of $\Delta t_B = \frac{5L}{c}$. Hence, the required speed of D in B's frame for these two events to be at the same position in D's frame is

$$v_{D,B} = \frac{\Delta x_B}{\Delta t_B} = \frac{L}{\frac{5L}{c}} = \frac{c}{5}$$

Since the events happen at the same position in D's frame, by definition, D sees the proper time interval. Working out the associated γ factor and finding the proper time gives

$$\gamma_{D,B} = \frac{1}{\sqrt{1-\left(\frac{1}{5}\right)^2}} = \frac{5}{2\sqrt{6}} \quad \Longrightarrow \quad \Delta t_D = \frac{\Delta t_B}{\gamma_{D,B}} = \frac{\frac{5L}{c}}{\frac{5}{2\sqrt{6}}} = \frac{2\sqrt{6}L}{c}$$

(iii) Applying the invariant interval in Equation (5), we have:

$$(\Delta s_A)^2 = (\Delta x_A)^2 - (c\Delta t_A)^2 = L^2 - (5L)^2 = -24L^2$$

$$(\Delta s_B)^2 = (\Delta x_B)^2 - (c\Delta t_B)^2 = L^2 - (5L)^2 = -24L^2$$

$$(\Delta s_C)^2 = (\Delta x_C)^2 - (c\Delta t_C)^2 = \left(\left(\frac{7L}{c}\right)\left(\frac{3c}{5}\right) + \frac{4}{5}L\right)^2 - (7L)^2 = -24L^2$$

$$(\Delta s_D)^2 = (\Delta x_D)^2 - (c\Delta t_D)^2 = 0 - \left(2\sqrt{6}L\right)^2 = -24L^2$$

All of them are the same, which verifies the validity of our answers. The invariant interval is a powerful tool for you to check your answers in special relativity – you should be obtaining the same value for all frames!

1.4.3 Causality

Einstein's postulates showed that *nothing* can travel faster than c. This also applies to the speed of transmission of information – information propagates with a finite speed that cannot exceed c.

When we talk about **causality,** we are interested in whether two events are **causally connected;** that is, whether one could have been the cause of another. The speed limit of information is a tool that can prove two events cannot be causally connected.

A necessary, but **not sufficient** condition for two events A and B to be causally connected is

$$\left|\frac{x_A - x_B}{t_A - t_B}\right| \leq c \tag{26}$$

This means that if the ratio is indeed $\leq c$, events A and B *might* be causally connected. However, if the ratio is $> c$, then we can conclude with full certainty that the events A and B cannot be causally connected. This ratio can only be used to **disprove** the causality of two events.

When two events are causally connected, the **order** in which they happen must be the same in all inertial frames. Conversely, if two events are not causally connected, there exists different frames where the order of the events can be swapped.

1.4.4 Relativistic Doppler Effects

There are two types of relativistic Doppler effects – the **longitudinal Doppler effect** and the **transverse Doppler effect.**

Recall from waves on the equation of a wave. We can hence express the equation of a light wave in frame S and S' as such (ignoring the phase):

$$\text{In frame } S\text{: } y = A\sin\left(kx - \omega t\right) \tag{27}$$

$$\text{In frame } S'\text{: } y' = A'\sin\left(k'x' - \omega' t'\right) \tag{28}$$

From the invariance of the speed of light and the wave dispersion relation,

$$c = \frac{\omega}{k} = \frac{\omega'}{k'} \tag{29}$$

Applying the LT for x and x', y and y', and t and t', we obtain

$$y' = y \quad \Longrightarrow \quad A\sin\left(kx - \omega t\right) = A'\sin\left(k'x' - \omega' t'\right) \quad \Longrightarrow \quad \ldots \quad \Longrightarrow \quad \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \tag{30}$$

or, writing it in terms of the regular frequency,

$$f' = f\sqrt{\frac{1-\beta}{1+\beta}} \tag{31}$$

where f is the frequency emitted by the source, and f' is the frequency observed by the observer moving towards the source at β (with respect to the source).

Equation (31) is known as the **longitudinal Doppler effect.** You can verify that that when $v \ll c$ (i.e. $\beta \ll 1$), Equation (31) reduces to the normal non-relativistic Doppler effect.

On the other hand, the **transverse Doppler effect** is much more complicated, and is not even part of the IPhO syllabus. You may read more about it here if you are interested.

Remark. If not stated, the relativistic Doppler effect usually refers to the longitudinal one. Anyway, you will be encountering the longitudinal one much more frequently.

1.4.5 Proper Acceleration

Acceleration in special relativity also works differently, as you would expect.

The **proper acceleration,** a_0, is defined as the acceleration of a particle in frame S', such that at time $t' + \mathrm{d}t'$, the particle is moving at speed $a_0\,\mathrm{d}t'$ relative to the frame it was in at time t'. This is the acceleration of a particle in its instantaneous rest frame at time t'.

Let's attempt to derive an expression containing a_0 from relativistic velocity addition. Let a be the acceleration of the particle in frame S, and suppose it has speed v in this frame. The new speed in the lab frame is given by the below two quantities, which are equivalent:

$$v + a\,\mathrm{d}t = \frac{v + a_0\,\mathrm{d}t'}{1 + \frac{va_0\mathrm{d}t'}{c^2}} \tag{32}$$

Employing a first-order approximation, we ignore all terms of order $(\mathrm{d}t')^2$ and above. Also, from time dilation, as t' is recorded in the frame of the particle where the "events" of recording are happening at the same position (the origin in the particle's frame), it is the proper time, so $\mathrm{d}t = \gamma\,\mathrm{d}t'$.

Hence, Equation (32) simplifies into the following:

$$v + a\,\mathrm{d}t \approx (v + a_0\,\mathrm{d}t')\left(1 - \frac{va_0\,\mathrm{d}t'}{c^2}\right) \approx v + a_0\,\mathrm{d}t' - \frac{v^2a_0\,\mathrm{d}t'}{c^2} \quad \Longrightarrow \quad a\,\mathrm{d}t = \frac{a_0}{\gamma^2}\,\mathrm{d}t' \tag{33}$$

$$\Longrightarrow \quad a_{\parallel} = \frac{a_0}{\gamma^3} \tag{34}$$

Equation (34) relates the proper acceleration to the lab-frame acceleration. Here, the proper acceleration is **parallel** to the particle's motion (along the x-axis).

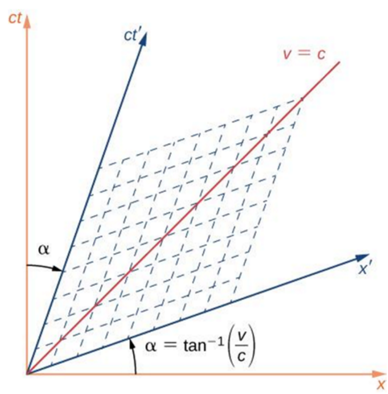
For transverse acceleration (i.e. acceleration **perpendicular** to the particle's velocity), we have instead (proof omitted):

$$a_{\perp} = \frac{a_0}{\gamma^2} \tag{35}$$

1.5 Minkowski Diagrams (And How To Read Them)

Minkowski diagrams are a graphical way of representing spacetime coordinates for a particular LT. Theoretically, you would need 4 axes (1 for time and 3 for position), but since nothing interesting happens in y and z, we only need 2 axes.

Most of the time, it is just a simple plot of ct against x, and ct' against x':



Every event can be represented by a point on this diagram. Afterwards, to perform the LT (or inverse LT), it reduces to reading off the desired coordinates in the desired frame.

Let's turn our attention to the point $(x', ct') = (0, 1)$, lying on the ct' axis. This point is 1 unit length in the ct' axis. Applying the inverse LT,

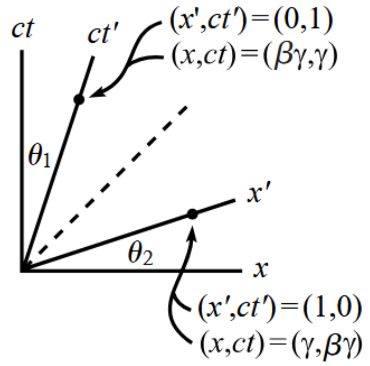
$$(x', ct') = (0, 1) \implies (x, ct) = (\gamma\beta, \gamma) \tag{36}$$

Hence, the angle between the ct and ct' axes (and also between the x and x' axes) is

$$\tan\theta = \frac{x}{ct} = \beta \tag{37}$$

and, the ratio between 1 unit on the ct and ct' axes (and also between the x and x' axes) is

$$\frac{1\,\text{ct' unit}}{1\,\text{ct unit}} = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \tag{38}$$



Of course, you can perfectly do every special relativity problem without this, but the Minkowski diagram is a nice tool that turns every special relativity problem into a geometry problem, if that's what you prefer.

1.6 Relativistic Dynamics

Now that we've covered relativistic kinematics, it's time to turn our attention to dynamics.

1.6.1 Relativistic Mass(?)

There is a theory that mass changes in special relativity. The **rest mass** is defined as the mass of an object in its rest frame, and is usually denoted as m_0.

The **relativistic mass** gives that the mass of a moving object increases, and is given by

$$m = \gamma m_0 \tag{39}$$

There are many caveats of using Equation (39). Physicists invented this to make explaining other relativistic dynamics concepts easier (as you shall see soon). However, many physicists strongly discourage using this, as it gets confusing (hence, the question mark).

Remark. If the type of mass is not specified, it is usually assumed that we are referring to the rest mass.

1.6.2 Total Energy, Kinetic Energy & Rest Energy

Einstein found that all massive objects at rest have energy, known as **rest energy** (sometimes referred to as mass-energy equivalence). The rest energy is given by

$$E_{\mathrm{R}} = m_0 c^2 \tag{40}$$

It was also found that the **kinetic energy** is given by

$$E_{\mathrm{K}} = (\gamma - 1)\, m_0 c^2 \tag{41}$$

which you can verify that for $\beta \ll 1$, a first-order expansion reduces to the usual classical $\frac{1}{2} m_0 v^2$.

The **total energy** of a particle is simply the sum of its kinetic energy and rest energy:

$$E_{\mathrm{T}} = E_{\mathrm{K}} + E_{\mathrm{R}} = \gamma m c^2 \tag{42}$$

You might not be convinced at all about Equations (40) to (42). The truth is, these are all empirically derived. Physicists in the past conducted many high-energy experiments, such as colliding high-energy subatomic particles, and observed that energies follow these relations. We simply accept these experimental claims because we realise that they reduce to classical results for low speeds, as expected.

1.6.3 Relativistic Momentum

The **relativistic momentum** of a massive particle is defined as

$$\mathbf{p} = \gamma m_0 \mathbf{v} \tag{43}$$

There is a very neat equation that ties up all the quantities we have covered so far, known as the **relativistic dispersion equation:**

$$E_{\mathrm{T}}^2 = E_{\mathrm{R}}^2 + p^2 c^2 = m_0^2 c^4 + p^2 c^2 \tag{44}$$

Remark. In special relativity, energy and momentum work just like in classical mechanics. Conservation laws such as COE and COM still exist. However, be sure to use the correct relativistic expressions!

In 1D, the following expressions still hold in special relativity:

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{45}$$

$$F = \frac{\mathrm{d}E}{\mathrm{d}x} \tag{46}$$

These are going to be very useful, since you'd want to work with energy and momentum most of the time in special relativity.

The next few examples on relativistic dynamics will be particularly useful.

Example 1.4 (USAPhO 2012, Problem A1, modified)**.** A newly discovered subatomic particle, the S meson, has a mass M. When at rest, it lives for some time before decaying into two identical particles called P mesons that each have a mass of αM. For the following parts, consider a reference frame where the S meson is at rest. (i) Determine the kinetic energy of each P meson particle. (ii) Determine the momentum of each P meson particle. (iii) Determine the velocity of each P meson particle.

(i) By COE, if the kinetic energy of each P meson particle is $E_{\mathrm{K,P}}$, we have

$$E_{\mathrm{R,S}} = 2E_{\mathrm{T,P}} = 2\left(E_{\mathrm{R,P}} + E_{\mathrm{K,P}}\right) \quad \Longrightarrow \quad Mc^2 = 2\left(\alpha Mc^2 + E_{\mathrm{K,P}}\right)$$

$$\Longrightarrow \quad E_{\mathrm{K,P}} = \left(\frac{1}{2} - \alpha\right) Mc^2$$

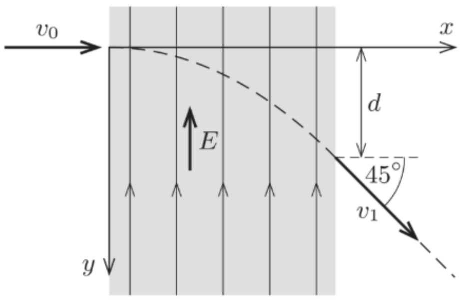
(ii) By symmetry, each P meson particle must have a total energy $E_{\mathrm{T,P}} = \frac{1}{2}E_{\mathrm{R,S}} = \frac{1}{2}Mc^2$. Hence, using the relativistic dispersion relation,

$$E_{\mathrm{T,P}}^2 = p_{\mathrm{P}}^2 c^2 + \alpha^2 M^2 c^4 \quad \Longrightarrow \quad p_{\mathrm{P}} = \sqrt{\frac{E_{\mathrm{T,P}}^2 - \alpha^2 M^2 c^4}{c^2}} = Mc\sqrt{\frac{1}{4} - \alpha^2}$$

(iii) We can relate the velocity of the P meson particle to its total energy and momentum:

$$p_{\mathrm{P}} = \gamma\alpha M v_{\mathrm{P}},\ E_{\mathrm{T,P}} = \gamma\alpha Mc^2 \quad \Longrightarrow \quad v_{\mathrm{P}} = \frac{p_{\mathrm{P}}c^2}{E_{\mathrm{T,P}}} = c\sqrt{1 - 4\alpha^2}$$

Example 1.5 (200 More Puzzling Physics Problems)**.** An electron moving with speed $v_0 = 0.6c$ enters a homogeneous electric field that is perpendicular to its initial velocity. When the electron leaves the field, its velocity makes an angle 45° with its initial direction. (i) Find the speed v_1 of the electron after it has exited the electric field. (ii) Find the distance d shown below, if the strength of the electric field is $E = 510\,\mathrm{kV/m}$.



(i) Let the velocity at any time be $\mathbf{v}$ with angle θ from the horizontal, as viewed in the lab frame. Then,

$$\mathbf{F} = -e\mathbf{E} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v}$$

Based on the defined coordinate system,

$$\mathbf{v} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix} \quad \Longrightarrow \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \dot{v}\cos\theta - v\dot{\theta}\sin\theta \\ \dot{v}\sin\theta + v\dot{\theta}\cos\theta \end{pmatrix}$$

It can be very helpful to first find the following:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{1-\beta^2}}\right) = \frac{-\frac{1}{2\sqrt{1-\beta^2}}\left(-2\beta\dot{\beta}\right)}{1-\beta^2} = \frac{\beta\dot{\beta}}{(1-\beta^2)^{\frac{3}{2}}} = \gamma^3\beta\dot{\beta}$$

Hence, we have a pair of ODEs:

$$-e\begin{pmatrix} 0 \\ E \end{pmatrix} = m\gamma\begin{pmatrix} \dot{v}\cos\theta - v\dot{\theta}\sin\theta \\ \dot{v}\sin\theta + v\dot{\theta}\cos\theta \end{pmatrix} + m\gamma^3\beta\dot{\beta}\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

It remains to solve this system of ODEs, which is left as an exercise. Eventually, you should get

$$v = 0.728c$$

(ii) Using COE, we simply have

$$eEd = \Delta\left(\gamma mc^2\right) = mc^2\Delta\gamma = mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}\right) \implies d = 20.8\,\text{cm}$$

1.7 Ideas

Many tricky special relativity problems involve the use of the following ideas.

1.7.1 Light Time

Light time is a very subtle thing that is often missed out, especially if you are careless.

Essentially, there is a difference between *observing* and *seeing*. When we say an observer observes an event, it refers to when the event actually happens. But, when we say an observer *sees* an event, we must account for the time taken for light to travel to their eyes! You need to be careful with the question's wording.

1.7.2 Rapidity

In special relativity, velocity addition as per Equation (23) appears to be a very tedious process, especially if you are going across many, many frames. As such, the **rapidity** is a neat mathematical trick to help you add velocities more efficiently.

The rapidity, ϕ, is defined using

$$\beta = \tanh\phi \implies \phi = \tanh^{-1}\beta \tag{47}$$

This is neat, because of the following identity:

$$\tanh(x+y) \equiv \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \tag{48}$$

Notice how this closely resembles velocity addition! This means that unlike velocities (or βs), **rapidities are linearly additive!** You can manipulate rapidities in a so-called Galilean manner.

This saves us a lot of trouble when we have to do many relativistic velocity additions, such as in the example below.

Example 1.6 (Ricardo)**.** Frame S_1 moves at β w.r.t frame S, frame S_2 moves at β w.r.t frame S_1, and so on. Find the velocity of frame S_{10} as seen by an observer in frame S.

If you didn't know about rapidity, you would have to do 10 relativistic velocity additions. Thankfully, with rapidity, we can add them up normally, and hence the final rapidity is

$$\phi_\text{f} = 10\phi = 10\tanh^{-1}\beta$$

As such, the desired velocity is

$$v_{10} = \beta_{10}c = c\tanh\left(10\tanh^{-1}\beta\right) = \left(\frac{(1+\beta)^{10} - (1-\beta)^{10}}{(1+\beta)^{10} + (1-\beta)^{10}}\right)c$$

where you can get the last equality if you know a little about hyperbolic trig identities.

As a check, when $\beta \ll 1$, $v_{10} = 10\beta c$, which is exactly what you'd expect if you added the velocities under Galilean relativity.

1.7.3 4-Vectors

A **4-vector** V^μ is a set of 4 quantities (V^0, V^1, V^2, V^3) that transform in the same manner as (ct, x, y, z) under the LT. It comprises of 1 "time-like" component V^0 and 3 "space-like" components V^1, V^2, V^3.

By definition, for V^μ to be a 4-vector, V^0 and V^1 must obey

$$\begin{pmatrix} V^{1'} \\ V^{0'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} V^1 \\ V^0 \end{pmatrix} \tag{49}$$

and we assume V^2 and V^3 are not transformed, as per the standard configuration.

The **inner product** of two 4-vectors V and W is written and defined as

$$V \cdot W := -V^0W^0 + V^1W^1 + V^2W^2 + V^3W^3 \tag{50}$$

This is basically the dot product but for 4-vectors. Sometimes, it is also written as $V^\mu W_\mu$.

Remark. You may see some places define the inner product as the negative of Equation (50). This is perfectly fine, as long as you are consistent with signs. But for this handout, we shall adopt the metric signature of $(-, +, +, +)$; that is, we put a negative for the product of the time-like components and leave the product of the space-like components as it is.

The elegant thing about the inner product is that the inner product of two 4-vectors is **invariant across all frames** (i.e. Lorentz invariant). That means, we can pick some convenient frame to perform the inner product in, and then move to a desired frame to do our calculations!

But what are some examples of 4-vectors? We define the **4-position** x^μ as

$$x^\mu = (ct, x, y, z) \tag{51}$$

Clearly, if we multiply (or divide) a 4-vector with another Lorentz invariant quantity, it will still stay Lorentz invariant. We know that the proper time τ is Lorentz invariant, hence we can define the **4-velocity** as

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = (\gamma c, \gamma v_\mathrm{x}, \gamma v_\mathrm{y}, \gamma v_\mathrm{z}) = (\gamma c, \gamma \mathbf{v}) \tag{52}$$

In the spirit of the 4-velocity, let's go one step further into the **4-acceleration.** For the case where $\mathbf{v} = v\hat{\mathbf{x}}$, we shall find an expression for a^μ. The 4-acceleration is defined as

$$a^\mu = \frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}v^\mu}{\mathrm{d}t} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\left(c, \mathbf{v}\right)\right) = \ \dots\ = \left(\frac{\gamma^4 a_\mathrm{x} v}{c}, \gamma^4 a_\mathrm{x}, \gamma^2 a_\mathrm{y}, \gamma^2 a_\mathrm{z}\right) \tag{53}$$

Example 1.7 (Ricardo)**.** (i) Find the square of the magnitude of the 4-velocity. (ii) For any two objects A and B, show that the inner product of their 4-velocities is given by $-c^2\gamma_\mathrm{rel}$, where v_rel is their relative velocity.

(i) We just take the inner product of the 4-velocity with itself:

$$|v^\mu|^2 = v^\mu v_\mu = -\gamma^2c^2 + \gamma^2v^2 = -\gamma^2\left(c^2 - v^2\right) = -c^2$$

Surprisingly, this means the square of the magnitude of the 4-velocity is always a constant!

(ii) We make use of how the inner product of two 4-vectors is Lorentz invariant. WLOG, go into A's frame. Then, the required inner product is

$$(c, 0, 0, 0) \cdot (\gamma_{\text{rel}} c, \gamma_{\text{rel}} v_{\text{rel}}, 0, 0) = -\gamma_{\text{rel}} c^2$$

There are also important 4-vectors in relativistic dynamics. The most important one is the **4-momentum,** which we define as

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right) = \left(\frac{E}{c}, p_{\text{x}}, p_{\text{y}}, p_{\text{z}}\right) \tag{54}$$

Since this is a 4-vector, it transforms like x^μ. This gives you a way to transform energy and momentum between frames!

Remark. For relativistic dynamics problems, it is *almost* always better to work in terms of momentum and energy instead of velocity and acceleration. You can see from Equation (53) how messy working with acceleration becomes!

1.7.4 Relativistic Collisions

Now that you've learnt about the 4-momentum, all relativistic collision problems reduce down to applying the **conservation of 4-momentum.** It is essentially a much cleaner way to express COE and COM in a single equation.

The way to apply it is as follows:

1. Start by writing out all the relevant initial and final 4-momentum expressions, to form the equation.
2. Identify the known and unknown 4-momenta. Isolate an unknown/an irrelevant 4-momentum term to one side of the equation.
3. Square both sides of the equation.

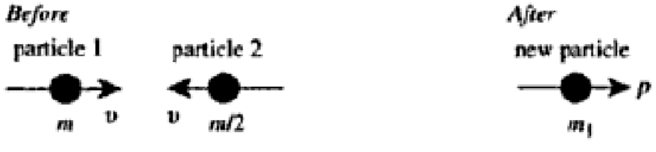
Squaring both sides is nice, because the inner product of any 4-momentum with itself gives

$$p^\mu p_\mu = -m^2 c^2 \tag{55}$$

which can be proven using Equation (44). For photons, it will be 0.

At this juncture, relativistic collisions might seem abstract or vague, but the next few examples will illustrate how to use 4-momentum to solve them.

Example 1.8 (Ricardo)**.** A particle of rest mass m moving along the x-axis with velocity v collides with a particle of rest mass $\frac{m}{2}$ moving along the x-axis with velocity $-v$. If the two particles coalesce, find the rest mass of the resulting particle.



The conservation of 4-momentum equation is

$$p_1^\mu + p_2^\mu = p^\mu \quad \Longrightarrow \quad \left(\frac{E_1}{c}, p_1, 0, 0\right) + \left(\frac{E_2}{c}, p_2, 0, 0\right) = \left(\frac{E}{c}, p, 0, 0\right)$$

Squaring both sides of the equation gives

$$-m^2c^2 - \left(\frac{m}{2}\right)^2 c^2 + 2\left(-\frac{E_1E_2}{c^2} + p_1p_2\right) = -m_1^2c^2$$

Recalling that $E_1 = \gamma mc^2, E_2 = \frac{1}{2}\gamma mc^2, p_1 = \gamma mv, p_2 = -\frac{1}{2}\gamma mv$ where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, we have

$$-\frac{5}{4}m^2c^2 + 2\left(-\frac{1}{2}\gamma^2m^2c^2 - \frac{1}{2}\gamma^2m^2v^2\right) = -m_1^2c^2 \implies -\frac{5}{4}m^2c^2 - \gamma^2m^2\left(c^2+v^2\right) = -m_1^2c^2$$

$$\implies -\frac{5}{4}m^2c^2 - \frac{m^2c^2\left(c^2+v^2\right)}{c^2-v^2} = -m_1^2c^2 \implies m_1 = m\sqrt{\frac{5}{4}+\frac{c^2+v^2}{c^2-v^2}} = \frac{m}{2}\sqrt{\frac{9c^2+v^2}{c^2-v^2}}$$

1.7.5 Should I Set $c = 1$?

In many textbooks and handouts, you may see seemingly dimensionally-incorrect expressions, such as $E_{\mathrm{T}} = \gamma m$, $v = \frac{p}{E_{\mathrm{T}}}$ and so on. These arise because it is common to work in units where $c = 1$ in special relativity, simply because c appears too much and is annoying to deal with. At the end, you then restore factors of c by checking dimensions.

The general advice is to **only set $c = 1$ if you are confident in what you are doing.** While your expressions become simpler to work with, you are unable to check your working via dimensional analysis. You have to decide what's best for yourself.

2 Problems

Problems are arranged in roughly increasing difficulty. Take $c = 3.00 \times 10^8$ m/s if needed.

Problem 2.1 (SPhO 2019). One event occurs at the origin of an inertial frame S at the time $t = 0$. Another event occurs at $x = 4c$, $y = z = 0$, $t = 5\,\text{s}$ relative to S. (i) Determine the velocity, relative to S, of the inertial frame S' in which the two events are recorded at the same point in space. (ii) What is the time interval between the events in the S' frame moving at the velocity found in (i) relative to the S frame?

Problem 2.2 (SPhO 2019). A cube with sides of length l is moving with one of its sides parallel to the x-axis of an inertial frame S with velocity u. An observer is moving along the x-axis of the inertial frame S with velocity v. Both u and v are comparable to c, the speed of light. Derive an expression for the volume of the cube as measured by the observer in terms of l, u, v and c.

Problem 2.3 (SPhO 2008). A flash of light is sent out from a point x_1 on the x-axis of an inertial frame S, and it is received at a point $x_2 = x_1 + l$. Consider another inertial frame S', moving with constant speed $v = bc$ along the x'-axis. (i) Show that, in S', the separation between the point of emission and the point of reception of the light is

$$l' = l\left(\frac{1-b}{1+b}\right)^{\frac{1}{2}}$$

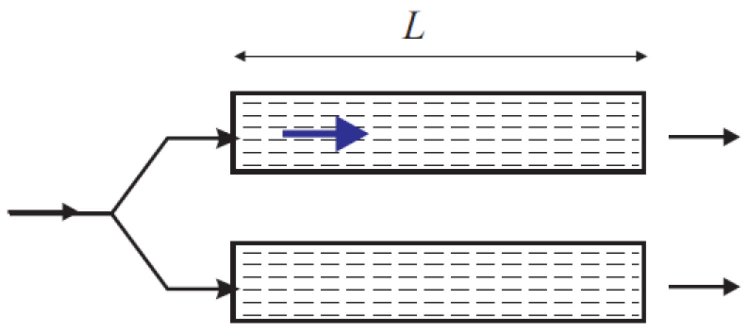
(ii) Show that, in S', the time interval between the emission and reception of light is

$$\Delta\, t' = \frac{l}{c}\left(\frac{1-b}{1+b}\right)^{\frac{1}{2}}$$

Problem 2.4 (SPhO 2009, 2011). A moving rod is observed to have a length of 2.00 m and oriented at an angle of 30.0° with respect to the direction of motion, as shown in the figure below. The rod has a speed of $0.995c$. (i) What is the proper length of the rod? (ii) What is the orientation angle in the proper frame? *Yes, the exact same question was repeated across two years.*

Problem 2.5 (SPhO 2018). A rocket of proper length 600 m is moving directly away from the Earth with uniform velocity. A radar pulse is sent out from the Earth and is reflected from the reflectors at the back end and the front end of the rocket. The first reflected radar pulse is received back at the base 5.00 min after emission and the second reflected pulse is received 12.0 μs later. (i) Calculate the distance of the rocket from the Earth at the instant the outgoing radar pulse hits the back end reflector. (ii) Calculate the velocity of the rocket relative to Earth. (iii) Calculate the time interval between the reflections at the back end and front end of the rocket measured in the inertial frame of the rocket. (iv) Explain why the time interval between the reflections in the two frames (i.e. the Earth frame and the rocket frame) are not related by the time dilation formula.

Problem 2.6 (SPhO 2006). A beam of monochromatic light, whose wavelength and speed in free space is λ and c respectively, is split into two separate beams and each of them is passed through identical troughs of water of length L. If the water in one trough is stationary and the water in the other trough is moving with speed $v \ll c$ in the direction of the light, determine the phase difference in terms of L, λ, v, c, n, where n is the refractive index of the stationary water.



Problem 2.7 (SPhO 2013, modified). A "perfect" rocket engine combines matter and anti-matter in a controlled way to yield photons, all of which are directed out of a spaceship. Suppose we start with a spaceship of initial mass M_0 at rest, and that after fully burning out, the remaining spaceship moves at speed v and has mass $m = fM_0$. You may take $c = 1$, and hence $\gamma = \frac{1}{\sqrt{1-v^2}}$. (i) What is the initial total energy of the system? Let E_{rad} stand for the total energy of radiation after the burnout. Find an expression for the total energy of the system after the burnout and set up the COE equation. (ii) Write down the COM equation. (iii) Show that $\gamma f + \gamma v f = 1$. (iv) Hence, show that $f^2 - 2\gamma f + 1 = 0$.

Problem 2.8 (SPhO 2004). In its rest frame, a source emits light in a conical beam of width $\pm 60°$. For a frame moving towards the source at high speed v, comparable to the speed of light c, the beam width is $\pm 45°$. Determine the value of v.

Problem 2.9 (SPhO 2005). Two particles with rest masses m_1 and m_2 move collinearly in some inertial frame with uniform velocities u_1 and u_2 respectively. They collide and form a single particle with rest mass m moving at velocity u. (i) Prove that

$$m^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1 u_2}{c^2}\right)$$

where $\gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. (ii) Find an expression for u.

Problem 2.10 (SPhO 2006). A photon of frequency f is scattered through an angle of $90°$ after colliding with an electron of mass m_e initially at rest. (i) Show that its frequency f' after being scattered is given by

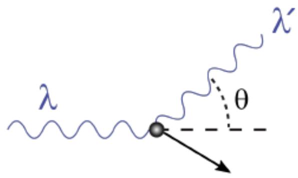
$$f' = \frac{m_e c^2}{hf + m_e c^2} f$$

where h is Planck's constant. (ii) Determine the kinetic energy of the recoiling electron after the scattering if the X-ray photon has initial energy 50.0 keV.

Problem 2.11 (SPhO 2016). Consider a spaceship initially at rest in the lab frame. At a given instant, it starts to accelerate with constant proper acceleration a_0. Assume that the lab clock t and the spaceship clock t' are synchronised such that this happens when $t = t' = 0$. (i) Determine the relative speed of the spaceship and the lab frame in terms of t'. Also, check the limit of the relative speed found when $a_0 t' \ll c$ and comment on the validity of the result. (ii) Find the relation between t and t'. Also, explore the relation when $a_0 t' \gg c$ and comment on the validity of the result.

Problem 2.12 (SPOT TST 2018)**.** An interstellar physics teacher travels at constant velocity v relative to Earth. As her rocket grazes past her class of Earth-bound students, she sends out a signal for them to begin their test. The teacher would like the class to have time T to complete the test. When should she send out a beam of light back to Earth in order to signal the end of the test?

Problem 2.13 (SPOT TST 2022)**.** A photon with wavelength λ scatters at an angle θ off an electron of mass m initially at rest, as shown in the figure below. Denote the wavelength of the scattered photon as λ'.



(i) Write down a relativistic expression for the electron energy E after the scattering event, in terms of its rest mass m, the magnitude of its momentum P and other relevant constants. (ii) By considering momentum conservation, show that

$$P^2 = h^2 \left(\frac{1}{\lambda^2} + \frac{1}{\lambda'^2} - \frac{2\cos\theta}{\lambda\lambda'} \right)$$

(iii) By also considering energy conservation, show that

$$\lambda' - \lambda = \lambda_C (1 - \cos\theta)$$

where λ_C is known as the Compton wavelength. Determine an expression for λ_C in terms of m and other relevant constants. (iv) Compton's original experiment was based on X-rays bombarding a graphite target. He found that some X-rays experienced no wavelength shift despite being scattered through large angles. Suggest an explanation for this.

Problem 2.14 (Kevin Zhou)**.** (i) Find the frequency of light reflected directly back from a mirror which is approaching the observer with speed v, if the light originally had frequency f. (ii) Show that this is the same as if the light were sourced with frequency f by an object moving at a speed of $\frac{2v}{1+\frac{v^2}{c^2}}$ towards the observer. Can you find an intuitive reason for this? (iii) Confirm that the energy gained by the light is equal to the loss of the kinetic energy of the mirror. For simplicity, assume that the mirror is very heavy and its mass doesn't change. (iv) Suppose the mirror lies at $x = 0$, the light that hits it travelling in the $\hat{\mathbf{x}}$ direction, and the mirror has a velocity $v\hat{\mathbf{y}}$. After reflection, which way does the light travel, and what is its new frequency?

Problem 2.15 (SPhL 2022)**.** A neutral atom of unknown rest mass m_n is travelling rightwards, towards a positive ion of unknown rest mass m_p and charge $+e$ (where e is the elementary charge) that is travelling leftwards. There is also an electric field of $E = 1.0\,\text{kV/m}$ pointing rightwards. Both particles collide with each other at the origin, with equal and opposite velocities of magnitude $v_0 = 0.5c$ at the instant of collision. After the collision, they stick together. The resulting composite particle moves leftwards for a while, before reversing and returning to the origin at a time $\Delta t = 15.0\,\text{ms}$ after the collision. If this composite particle is determined to have total (relativistic) energy $E_\text{T} = 30.0\,\text{GeV}$ upon returning to the origin, find (i) the original rest mass of the neutral atom m_n and (ii) the original rest mass of the positive ion m_p. Leave your answers in terms of u (the atomic mass unit).

Problem 2.16 (Morin)**.** A square with side length L flies past you at speed v, in a direction parallel to two of its sides. You stand in the plane of the square. When you **see** the square at its nearest point to you, show that it **looks** to you like it's rotated instead of contracted. (Assume that L is small compared to the distance between you and the square.)

Problem 2.17 (Kevin Zhou)**.** Consider a particle at the origin at time $t = 0$, with initial x-momentum p_0 and total energy E_0. A constant force F acts on the particle in the $-y$ direction. (i) Find $y(t)$. *Hint: Don't write down any equations containing* γ*, because it depends on* $v_x(t)$*, which we don't know yet.* (ii) Find $x(t)$. (iii) Combine these results to get $y(x)$. This is the path of a relativistic projectile.

Problem 2.18 (USAPhO 2021, Problem A2)**.** An excellent problem that generalises the relativistic Doppler effect.

Problem 2.19 (USAPhO 2023, Problem B2, modified)**.** A spaceship of mass m is propelled by light produced by lasers on Earth, with total power P. The light evenly impacts a sail on the spaceship, and reflects directly backwards. If the spaceship starts near Earth at rest, how long will it take, in the Earth's frame, to accelerate the spaceship to a speed v_{f}? You may leave your answer in terms of a definite integral in v (the integral is difficult to evaluate).

3 Advanced Problems

These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1 (EuPhO 2024, T2)**.** An excellent relativistic kinematics problem.

Problem 3.2 (IPhO 2022, T3C)**.** An excellent problem on proper acceleration and scaling laws.

Problem 3.3 (APhO 2013, T2)**.** One of the best special relativity problems out there, encompassing basically every concept that you need to know for Physics Olympiad.